

# Quotients of anti-de Sitter space

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## Abstract

We study the quotients of  $n + 1$ -dimensional anti-de Sitter space by one-parameter subgroups of its isometry group  $SO(2, n)$  for general  $n$ . We classify the different quotients up to conjugation by  $O(2, n)$ . We find that the majority of the classes exist for all  $n \geq 2$ . There are two special classes which appear in higher dimensions: one for  $n \geq 3$  and one for  $n \geq 4$ . The description of the quotient in the majority of cases is thus a simple generalisation of the  $AdS_3$  quotients.

The study of the propagation of strings on more general curved backgrounds is important both because it allows us to confront some of the important problems arising in any theory of quantum gravity (such as the problem of time), and because describing strings on time-dependent backgrounds is essential to address the phenomenological application of string theory to cosmology. A new class of simple supersymmetric backgrounds referred to as null branes was recently constructed [1], by considering a novel class of Kaluza-Klein reductions of flat space. These do not have a timelike Killing field, so they provide interesting examples for studying string theory on more general backgrounds; in addition, a subclass of ‘parabolic orbifolds’ have initial singularities. String theory on these backgrounds has been intensively studied, to expand our understanding of string theory in non-static backgrounds and to attempt to gain insight into the resolution of such spacetime singularities in string theory [2, 3, 4]. Unfortunately, unlike in more familiar spacelike orbifolds, it turns out that the singular geometries suffer from an instability, so the resolution of the singularities is not accessible in perturbation theory [5, 3, 4, 6].

It is natural for many reasons to wish to extend these investigations to consider strings on orbifolds of Anti-de Sitter space (AdS). First, AdS is also a maximally symmetric space, so it has a large isometry group which can lead to interesting examples of quotients. Secondly, the AdS/CFT correspondence [7, 8] provides a non-perturbative definition of string theory, which may enable us to obtain more insight into issues such as singularity resolution in an AdS context. Finally, it is well-known that a

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black hole geometry can be constructed from a quotient of  $\text{AdS}_3$  [9, 10]. These constructions therefore also offer an opportunity to explore backgrounds with non-trivial causal structure.

Such an extension was initiated in [11], where an AdS version of the isometry involved in the null brane quotient was constructed. Our aim in the present paper is to make a more systematic investigation of this question, classifying all the physically distinct quotients of  $\text{AdS}_{n+1}$  by one-parameter subgroups of its isometry group. The classification of quotients of  $\text{AdS}_3$  was thoroughly explored in [12]. This was extended to  $\text{AdS}_4$  in [13]. Our aim is to extend this to general dimensions, and in particular to address the case of  $\text{AdS}_5$ , of great interest for string theory. This question has also been explored independently by Figueroa-O'Farrill and Simon [14], who also investigate quotients with a non-trivial action on the sphere factor in  $\text{AdS}_p \times S^q$  backgrounds in string theory and investigate the supersymmetry preserved under their quotients.

We will show that the classification of physically distinct one-parameter subgroups of  $SO(2, n)$  extends very naturally from the case  $n = 2$  to higher  $n$ . The subgroups considered in [12] all have higher-dimensional generalisations, whose analysis is directly related to the analysis in the case of  $\text{AdS}_3$ . There are only two further physically distinct possibilities, one of which appears for all  $n \geq 3$ , and the other of which appears for all  $n \geq 4$ . The prototype example of the former was discussed in [13], and the latter contains the null brane-like quotient discussed in [11].

The purpose of this paper is to describe the basic steps in the classification of the quotients and the construction of normal forms for the Killing vectors in some detail. We will also briefly explore how the coordinate systems can be adapted to directly relate higher-dimensional quotients to lower-dimensional ones, but we postpone detailed exploration of the physics of these quotients to a companion paper with Figueroa-O'Farrill and Simon [15].

We wish to classify quotients of  $\text{AdS}_{n+1}$  by one-parameter subgroups of  $SO(2, n)$ .<sup>1</sup> A one-parameter subgroup is determined by a Killing vector  $\xi^\mu$  in the Lie algebra  $so(2, n)$ ; such a Killing vector can be written in terms of a basis  $J_{ab}^\mu$  of  $so(2, n)$  as  $\xi = \omega^{ab} J_{ab}$ , where  $\omega^{ab} = -\omega^{ba}$ . If we describe  $\text{AdS}_{n+1}$  in terms of embedding coordinates  $U, V, X_i$  ( $i = 2, \dots, n+1$ ) such that  $-U^2 - V^2 + X_i^2 = -1$ , then the  $J_{ab}$  are

$$J_{01} = V\partial_U - U\partial_V, \quad J_{0i} = U\partial_i + X_i\partial_U, \quad J_{1i} = V\partial_i + X_i\partial_V, \quad J_{ij} = X_i\partial_j - X_j\partial_i. \quad (1)$$

The classification of physically different  $\xi^\mu$  is therefore equivalent to classifying antisymmetric matrices  $\omega^{ab}$  up to conjugation by elements of  $SO(2, n)$ , that is,  $\omega' \sim \omega$  iff  $\omega'^a_b = (T^{-1})^a_c \omega^c_d T^d_b$  for some  $T^a_c \in SO(2, n)$ . As explained in [12, 13], if we slightly extend the equivalence relation, so that  $\omega' \sim \omega$  for  $T^a_c \in O(2, n)$ , then the problem is equivalent to the familiar problem of classifying the matrices up to similarity.

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<sup>1</sup>We will generally have in mind the quotient by a discrete subgroup, to construct another  $n+1$ -dimensional spacetime; the prototypical example is the BTZ black hole [16, 12]. It is also interesting to consider the Kaluza-Klein reduction along such a direction to construct an  $n$ -dimensional spacetime. For the purposes of classification, we can treat these two kinds of quotients together.

The distinct matrices are then classified by their eigenvalues and the dimensions of the irreducible invariant subspaces associated with them. This extension of the equivalence relation implies that we will not distinguish between Killing vectors which differ by a sign reversal of some of the embedding coordinates.

Since the classification reduces to the study of the eigenvalues and eigenspaces of the matrix  $\omega^a_b$ , we can ‘build up’ the general matrix from the different eigenspaces. We will therefore first consider the different possibilities for invariant subspaces consistent with the signature of spacetime, and then use these possible invariant subspaces as building blocks to construct all the possible inequivalent matrices  $\omega_{ab}$ , and hence classify the different quotients. In the following we shall say that the matrix  $\omega_{ab}$  is of type  $k$  if its highest dimensional irreducible invariant subspace is of dimension  $k$ .

The calculations are simplified by observing that as a consequence of the fact that  $\omega_{ab}$  is real and antisymmetric, its eigenvalues come in groups: if  $\lambda$  is an eigenvalue of  $\omega^a_b$  then  $-\lambda$  is an eigenvalue of  $\omega^a_b$ , and similarly if  $\lambda$  is an eigenvalue then so is  $\lambda^*$ . Another useful fact is that if  $v^a$  and  $u^a$  are eigenvectors of  $\omega^a_b$  with respective eigenvalues  $\lambda$  and  $\mu$ , so that

$$\omega^a_b v^b = \lambda v^a, \quad \omega^a_b u^b = \mu u^a, \quad (2)$$

then  $v^a u_a = 0$  unless  $\lambda + \mu = 0$ . Note that  $v^a$  etc. are vectors in  $\mathbb{R}^{2,n}$ ; the indices on  $\omega_{ab}$ ,  $v^a$  etc are raised and lowered with the metric  $\eta_{ab}$  on  $\mathbb{R}^{2,n}$ . Thus, we see that  $\mathbb{R}^{2,n}$  decomposes into a product of orthogonal eigenspaces, but each such subspace is associated not with a single eigenvalue  $\lambda$  but with the pair of eigenvalues  $\lambda, -\lambda$ . We will now study the properties of these orthogonal eigenspaces.

Let us first discuss the cases with non-degenerate eigenvalues. The simplest case is when the eigenvalue is zero; then there is a single eigenvector  $v^a$ , which is orthogonal to all other eigenvectors, and by the non-degeneracy of the metric must then have  $v^a v_a \neq 0$ . We can rescale  $v^a$  to set  $v^a v_a = 1$ , which we will refer to as  $\lambda_0(+)$ , or  $v^a v_a = -1$ , which we will refer to as  $\lambda_0(-)$ . These cases correspond physically to a direction in  $\mathbb{R}^{2,n}$  which is not affected by the identification.

The next possibility is a pair of real eigenvalues,  $a, -a$ ,  $a \geq 0$ . Then we have

$$\omega_{ab} l^b = a l_a, \quad \omega_{ab} m^b = -a m_a. \quad (3)$$

The only non-zero inner product is  $l_a m^a = 1$ . To construct an orthonormal basis, we take

$$v_1 = \frac{1}{\sqrt{2}}(l + m), \quad v_2 = \frac{1}{\sqrt{2}}(l - m). \quad (4)$$

We then have  $v_1 \cdot v_1 = 1$ ,  $v_2 \cdot v_2 = -1$ , so this subspace has signature  $(-+)$ . We denote this by  $\lambda_r$ ; it corresponds physically to a boost in some  $\mathbb{R}^{1,1}$  subspace of  $\mathbb{R}^{2,n}$ .

If we have a pair of imaginary eigenvalues,

$$\omega_{ab} k^b = i b k_a, \quad \omega_{ab} k^{*b} = -i b k_a^*, \quad (5)$$

$b \geq 0$ , the only non-zero inner product is  $k_a k^{*a} = 1$ . Now we need to construct the orthonormal basis in a slightly different way, because we need to respect the fact that

the action of  $\omega_{ab}$  on  $\mathbb{R}^{2,n}$  is real-valued. We can set

$$v_1 = \frac{1}{\sqrt{2}}(k + k^*), \quad v_2 = \frac{i}{\sqrt{2}}(k - k^*). \quad (6)$$

We then have  $\omega_b^a v_1^b = b v_2^a$ ,  $\omega_b^a v_2^b = -b v_1^a$ . We have  $v_1 \cdot v_1 = 1$ ,  $v_2 \cdot v_2 = 1$ , so this subspace has signature  $(++)$ , which we denote by  $\lambda_i(++)$ . On the other hand, we could have chosen

$$v_1 = \frac{i}{\sqrt{2}}(k + k^*), \quad v_2 = \frac{1}{\sqrt{2}}(k - k^*). \quad (7)$$

This also gives a real action, but now  $v_1 \cdot v_1 = -1$ ,  $v_2 \cdot v_2 = -1$ , so this subspace has signature  $(--)$ , which we denote by  $\lambda_i(--)$ . These two cases correspond physically to rotations in  $\mathbb{R}^2$  subspaces of  $\mathbb{R}^{2,n}$ .

The final possibility is a complex eigenvalue, which gives us the four eigenvalues  $\lambda, -\lambda, \lambda^*, -\lambda^*$  (so we can take  $\lambda = a + ib$  for  $a, b \geq 0$ ). We have

$$\omega_{ab} l^b = \lambda l_a, \quad \omega_{ab} m^b = -\lambda m_a, \quad (8)$$

$$\omega_{ab} l^{*b} = \lambda^* l_a^*, \quad \omega_{ab} m^{*b} = -\lambda^* m_a^*. \quad (9)$$

The non-vanishing inner products are  $l \cdot m = 1$  and  $l^* \cdot m^* = 1$ , so  $l, m$  and  $l^*, m^*$  span two orthogonal two-dimensional spaces; however, we need to mix them to obtain a real basis. If we define

$$v_1 = \frac{1}{2}[(l + l^*) + (m + m^*)], \quad v_2 = \frac{1}{2}[(l + l^*) - (m + m^*)], \quad (10)$$

$$v_3 = \frac{i}{2}[(l - l^*) + (m - m^*)], \quad v_4 = \frac{i}{2}[(l - l^*) - (m - m^*)], \quad (11)$$

Then we will see that  $\omega_{ab}$  acts on the  $v_i$  with real coefficients, and they span a space of signature  $(- - ++)$ , which we denote  $\lambda_c$ .

Now we turn to the higher-dimensional invariant subspaces. If we have a  $k$ -dimensional subspace associated to the eigenvalue zero, then we can pick a basis of vectors  $m_i$ ,  $i = 1, \dots, k$  such that

$$\omega_{ab} m_1^b = 0, \quad \omega_{ab} m_i^b = m_{(i-1)a} \text{ for } i \neq 1. \quad (12)$$

We can then observe that  $m_1^a m_{(i-1)a} = m_1^a \omega_{ab} m_i^b = 0$  for  $i = 1, \dots, k$ . We then need  $m_1^a m_{ka} \neq 0$  for consistency with the non-degenerate metric. We can also use (12) to show

$$m_{ia} m_j^a = m_{ia} \omega^{ab} m_{(j+1)b} = -m_{(i-1)a} m_{(j+1)}^a, \quad (13)$$

and

$$m_{ia} m_{(i-1)}^a = m_{ia} \omega^{ab} m_{ib} = 0 \quad (14)$$

by antisymmetry of  $\omega_{ab}$ . Now imagine  $k$  is even. Then these two relations taken together imply that

$$m_{ka} m_1^a = m_{(k/2)a} m_{(k/2+1)}^a = 0, \quad (15)$$

in contradiction with the non-degeneracy of the metric. Hence there cannot be  $k$ -dimensional invariant subspaces associated with a zero eigenvalue for  $k$  even. For  $k$  odd, (13) implies

$$m_{ia}m_j^a = (-1)^{i+1}m_{1a}m_k^a \quad (16)$$

for  $i + j = k + 1$ . We can also set all other inner products to zero by a suitable redefinition of the basis  $m_i^a$ . We can then define an orthonormal basis by

$$v_{2i-1} = \frac{1}{\sqrt{2}}(m_i + m_{k+1-i}), \quad v_{2i} = \frac{1}{\sqrt{2}}(m_i - m_{k+1-i}) \text{ for } i = 1, \dots, \frac{k-1}{2}, \quad (17)$$

and  $v_k = m_{k+1/2}$ . We then have  $v_{2i-1} \cdot v_{2i-1} = -v_{2i} \cdot v_{2i}$ , and we can choose  $v_k \cdot v_k$  to be  $\pm 1$ , so the subspace spanned by these vectors has either  $(k-1)/2$  negative signature directions and  $(k+1)/2$  positive signature ones, or  $(k+1)/2$  negative signature directions and  $(k-1)/2$  positive signature ones. The only possibilities which are consistent with embedding as a subspace in  $\mathbb{R}^{2,n}$  are  $\lambda_0^{III}(-++)$  and  $\lambda_0^{III}(- - +)$ , and  $\lambda_0^V$  with signature  $(- - + +)$ .  $\lambda_0^{III}$  corresponds to a null rotation in an  $\mathbb{R}^{1,2}$  subspace of  $\mathbb{R}^{2,n}$ .

If we have a  $k$ -dimensional invariant subspace with a real eigenvalue, we must have a pair of them; we can define a basis such that the action of  $\omega_{ab}$  is

$$\omega_{ab}l_1^b = al_{1a}, \quad \omega_{ab}l_i^b = al_{ia} + l_{(i-1)a} \text{ for } i = 2, \dots, k, \quad (18)$$

and

$$\omega_{ab}m_1^b = -am_{1a}, \quad \omega_{ab}m_i^b = -am_{ia} + m_{(i-1)a} \text{ for } i = 2, \dots, k. \quad (19)$$

By repeatedly using these relations, we can show that  $l_i \cdot l_j = 0$  and  $m_i \cdot m_j = 0$  for all  $i, j$ . We can also show  $m_1 \cdot l_i = 0$  for  $i \neq k$ ; we then need  $m_1 \cdot l_k \neq 0$  for non-degeneracy. As in the case of a zero eigenvalue, we learn that

$$m_i \cdot l_j = (-1)^{i+1}m_1 \cdot l_k, \quad (20)$$

for  $i + j = k + 1$ , and we can set all other inner products to zero by a suitable redefinition of the basis. An orthonormal basis is then formed by taking

$$v_{2i-1} = \frac{1}{\sqrt{2}}(l_i + m_{k+1-i}), \quad v_{2i} = \frac{1}{\sqrt{2}}(l_i - m_{k+1-i}) \text{ for } i = 1, \dots, k. \quad (21)$$

We then have  $v_{2i-1} \cdot v_{2i-1} = -v_{2i} \cdot v_{2i}$ , so the subspace spanned by these vectors has an equal number of negative and positive signature directions. The only possibility consistent with being a subspace of  $\mathbb{R}^{2,n}$  is  $\lambda_r^{II}$ , which has signature  $(- - ++)$ .

If we have a  $k$ -dimensional invariant subspace with an imaginary eigenvalue, we must again have a pair of them; we can define a basis such that the action of  $\omega_{ab}$  is

$$\omega_{ab}k_1^b = ibk_{1a}, \quad \omega_{ab}k_i^b = ibk_{ia} + k_{(i-1)a} \text{ for } i = 2, \dots, k, \quad (22)$$

and

$$\omega_{ab}k_1^{*b} = -ibk_{1a}^*, \quad \omega_{ab}k_i^{*b} = -ibk_{ia}^* + k_{(i-1)a}^* \text{ for } i = 2, \dots, k. \quad (23)$$

By repeatedly using these relations, we can show that  $k_i \cdot k_j = 0$  and  $k_i^* \cdot k_j^* = 0$  for all  $i, j$ . We can also show  $k_1 \cdot k_i^* = 0$  for  $i \neq k$ ; we then need  $k_1 \cdot k_k^* \neq 0$  for non-degeneracy. As in the case of a zero eigenvalue, we learn that

$$k_i \cdot k_j^* = (-1)^{i+1} k_1 \cdot k_k^* \quad (24)$$

for  $i + j = k + 1$ , and we can set all other inner products to zero by a suitable redefinition of the basis. The action of  $\omega$  becomes real if we define new vectors  $w_i = \frac{1}{\sqrt{2}}(k_i + k_i^*)$  and  $x_i = \frac{i}{\sqrt{2}}(k_i - k_i^*)$ . There is then a technical difference between even and odd dimensions: in even dimensions, the non-zero inner products are  $w_i \cdot x_j$  for  $i + j = k + 1$ , and an orthonormal basis is formed by taking

$$v_{2i-1} = \frac{1}{\sqrt{2}}(w_i + x_{k+1-i}), \quad v_{2i} = \frac{1}{\sqrt{2}}(w_i - x_{k+1-i}) \text{ for } i = 1, \dots, k, \quad (25)$$

We then have  $v_{2i-1} \cdot v_{2i-1} = -v_{2i} \cdot v_{2i}$ . Thus, in even dimensions, we have a subspace with an equal number of positive and negative directions, and the only possibility in  $\mathbb{R}^{2,n}$  is  $\lambda_i^{II}$ , which has signature  $(- - ++)$ . In odd dimensions, the non-zero inner products are  $w_i \cdot w_j = x_i \cdot x_j$  for  $i + j = k + 1$ , and an orthonormal basis is formed by

$$v_{2i-1} = \frac{1}{\sqrt{2}}(w_i + w_{k+1-i}), \quad v_{2i} = \frac{1}{\sqrt{2}}(w_i - w_{k+1-i}) \text{ for } i = 1, \dots, \frac{k-1}{2}, \quad (26)$$

$$v_k = w_{\frac{k+1}{2}}, \quad v_{k+1} = x_{\frac{k+1}{2}} \quad (27)$$

$$v_{2i-1} = \frac{1}{\sqrt{2}}(x_i + x_{k+1-i}), \quad v_{2i} = \frac{1}{\sqrt{2}}(x_i - x_{k+1-i}) \text{ for } i = \frac{k+3}{2}, \dots, k. \quad (28)$$

We then have  $v_{2i-1} \cdot v_{2i-1} = -v_{2i} \cdot v_{2i}$  except for  $i = \frac{k+1}{2}$ ;  $v_k \cdot v_k = v_{k+1} \cdot v_{k+1}$ . The subspace thus either has  $k - 1$  positive and  $k + 1$  negative directions or vice-versa. The only possibility in  $\mathbb{R}^{2,n}$  is  $\lambda_i^{III}$ , which has signature  $(- - + + +)$ . In the special case  $b = 0$ , which will be important later,  $\lambda_i^{III}$  reduces to a pair of  $\lambda_0^{III}(- + +)$ —that is, to a pair of null rotations in independent subspaces. Finally, we could consider invariant subspaces of dimension  $k$  associated with complex eigenvalues. We will not give the details here, as it does not lead to any cases that fit inside  $\mathbb{R}^{2,d}$ . The subspace associated with the set of four complex eigenvalues always has at least  $2k$  negative directions.

In summary, the possible invariant subspaces and their signatures that can occur in our  $\omega_{ab}$  are  $\lambda_0(+)$ ,  $\lambda_0(-)$ ,  $\lambda_r(-+)$ ,  $\lambda_i(++)$ ,  $\lambda_i(--)$ ,  $\lambda_c(- - ++)$ ,  $\lambda_0^{III}(- + +)$ ,  $\lambda_0^{III}(- - +)$ ,  $\lambda_0^V(- - + + +)$ ,  $\lambda_r^{II}(- - + +)$ ,  $\lambda_i^{II}(- - + +)$ , and  $\lambda_i^{III}(- - + + +)$ .<sup>2</sup> Now let us consider how we can assemble these to form an  $n + 2$  dimensional matrix  $\omega_{ab}$ . For  $n$  even (which includes the case  $n = 4$  which we are particularly interested in), the possibilities are

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<sup>2</sup>Naturally, the same classification can be applied for the Lorentz group  $SO(1, n)$  in  $\mathbb{R}^{1,n}$ ; in that case, the only possible subspaces are  $\lambda_0(+)$ ,  $\lambda_0(-)$ ,  $\lambda_r(-+)$ ,  $\lambda_i(++)$ , and  $\lambda_0^{III}(- + +)$ , corresponding to trivial directions, boosts, rotations and null rotations respectively.

- Type *I*

$$\begin{aligned}
\mathbb{C} \quad & \lambda_c (- - ++ ) \quad + \frac{n-2}{2} \lambda_i (+_1 +_2 \cdots +_{n-2}), \\
\mathbb{R} \quad & 2\lambda_r (- - ++ ) \quad + \frac{n-2}{2} \lambda_i (+_1 +_2 \cdots +_{n-2}), \\
\mathbb{I} \quad & \frac{n+2}{2} \lambda_i (- - +_1 +_2 \cdots +_n).
\end{aligned}$$

Where the coefficient in front of a  $\lambda$  corresponds to the number of times that type of eigenvalue appears.

- Type *II*

$$\begin{aligned}
\mathbb{R} \quad & \lambda_r^{II} (- - ++ ) \quad + \frac{n-2}{2} \lambda_i (+_1 +_2 \cdots +_{n-2}), \\
\mathbb{I} \quad & \lambda_i^{II} (- - ++ ) \quad + \frac{n-2}{2} \lambda_i (+_1 +_2 \cdots +_{n-2}).
\end{aligned}$$

- Type *III*

$$\begin{aligned}
\mathbb{I} \quad & \lambda_i^{III} (- - + + + ) \quad + \frac{n-4}{2} \lambda_i (+_1 +_2 \cdots +_{n-4}), \\
0 \quad (a) \quad & \lambda_0^{III} (- + + ) \quad + \lambda_0 (+) \quad + \lambda_r (- + ) \quad + \frac{n-4}{2} \lambda_i (+_1 +_2 \cdots +_{n-4}), \\
0 \quad (b) \quad & \lambda_0^{III} (- - + ) \quad + \lambda_0 (+) \quad + \frac{n-2}{2} \lambda_i (+_1 +_2 \cdots +_{n-2}), \\
0 \quad (c) \quad & \lambda_0^{III} (- + + ) \quad + \lambda_0 (-) \quad + \frac{n-2}{2} \lambda_i (+_1 +_2 \cdots +_{n-2}).
\end{aligned}$$

- Type *V*

$$\lambda_0^V (- - + + + ) \quad + \lambda_0 (+) \quad + \frac{n-4}{2} \lambda_i (+_1 +_2 \cdots +_{n-4}).$$

To discuss the physics of these different cases, we need a convenient representative of each case. It is easy to construct suitable representatives; in most cases, this is a minor generalisation of the analysis of [12, 13], so we will just quote the result by giving the relevant Killing vectors. For  $I_{\mathbb{C}}$  this is

$$\xi = b_1(J_{01} + J_{23}) - a(J_{03} + J_{12}) + b_2 J_{45} + b_3 J_{67} + \cdots + b_{\frac{n}{2}} J_{nn+1}, \quad (29)$$

with  $a, b_i \geq 0$ .<sup>3</sup> The norm of this Killing vector is

$$\begin{aligned}
\xi_\mu \xi^\mu = & (a^2 - b_1^2)(X_{n+1}^2 + X_n^2 + \cdots + X_4^2 + 1) - 4ab_1(VX_3 - UX_2) \\
& + b_2^2(X_4^2 + X_5^2) + b_3^2(X_6^2 + X_7^2) + \cdots + b_{\frac{n}{2}}^2(X_n^2 + X_{n+1}^2). \quad (30)
\end{aligned}$$

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<sup>3</sup>Recall that we have identified Killing vectors differing by conjugation by  $O(2, n)$ ; if we only identified under conjugation by  $SO(2, n)$ , we should take  $b_i, i \geq 2$  to run over the reals, and  $\xi = b_1(-J_{01} + J_{23}) - a(-J_{03} + J_{12}) + b_2 J_{45} + b_3 J_{67} + \cdots + b_{\frac{n}{2}} J_{nn+1}$  and  $\xi = b_1(-J_{01} + J_{23}) - a(J_{03} - J_{12}) + b_2 J_{45} + b_3 J_{67} + \cdots + b_{\frac{n}{2}} J_{nn+1}$  for  $a, b_1 \geq 0$  would also count as distinct cases. Similar remarks apply in the other cases to follow.

Thus, this Killing vector can be everywhere spacelike for  $b_1 = 0$ . For type  $I_{\mathbb{R}}$  we have

$$\xi = a_1 J_{03} + a_2 J_{12} + b_1 J_{45} + \cdots + b_{\frac{n-2}{2}} J_{nn+1}, \quad (31)$$

with norm

$$\xi_\mu \xi^\mu = a_1^2 (U^2 - X_3^2) + a_2^2 (V^2 - X_2^2) + b_1^2 (X_4^2 + X_5^2) + \cdots + b_{\frac{n-2}{2}}^2 (X_n^2 + X_{n+1}^2). \quad (32)$$

This is everywhere spacelike for  $a_1 = a_2$  (using  $\eta^{ab} X_a X_b = -1$ ), which is equivalent to type  $I_{\mathbb{C}}$  with  $b_1 = 0$ . For type  $I_{\mathbb{I}}$  we have

$$\xi = b_1 J_{01} + b_2 J_{23} + b_3 J_{45} + \cdots + b_{\frac{n+2}{2}} J_{nn+1}, \quad (33)$$

with norm

$$\xi_\mu \xi^\mu = b_1^2 (-1 - X_2^2 - \cdots - X_{n+1}^2) + b_2^2 (X_2^2 + X_3^2) + b_3^2 (X_4^2 + X_5^2) + \cdots + b_{\frac{n+2}{2}}^2 (X_n^2 + X_{n+1}^2). \quad (34)$$

For  $b_1 = 0$ , this is spacelike away from the axis  $X_i = 0, i \geq 2$ , where the Killing vector degenerates, so this axis is a line of fixed points. For type  $II_{\mathbb{R}}$  we have

$$\xi = a(J_{03} + J_{12}) + J_{01} - J_{02} - J_{13} + J_{23} + b_1 J_{45} + \cdots + b_{\frac{n-2}{2}} J_{nn+1}, \quad (35)$$

with norm

$$\begin{aligned} \xi_\mu \xi^\mu = & a^2 (U^2 + V^2 - X_2^2 - X_3^2) + 4a(U - X_3)(X_2 + V) \\ & + b_1^2 (X_4^2 + X_5^2) + \cdots + b_{\frac{n-2}{2}}^2 (X_n^2 + X_{n+1}^2). \end{aligned} \quad (36)$$

For  $a = 0$ , this is spacelike except on the subspace  $X_i = 0, i \geq 4$ , where the Killing vector is null. For type  $II_{\mathbb{I}}$ , we have

$$\xi = (b_1 - 1)J_{01} + (b_1 + 1)J_{23} + J_{02} - J_{13} + b_2 J_{45} + \cdots + b_{\frac{n}{2}} J_{nn+1}, \quad (37)$$

with norm

$$\begin{aligned} \xi_\mu \xi^\mu = & b_1^2 (-1 - X_4^2 - \cdots - X_{n+1}^2) \\ & + 2b_1 (U + X_3)^2 + 2b_1 (V + X_2)^2 \\ & + b_2^2 (X_4^2 + X_5^2) + \cdots + b_{\frac{n-2}{2}}^2 (X_n^2 + X_{n+1}^2). \end{aligned} \quad (38)$$

For  $b_1 = 0$ , this is the same as type  $II_{\mathbb{R}}$  with  $a = 0$  (as one would expect). For type  $III_{\mathbb{I}}$  we have

$$\xi = b(J_{01} + J_{23} + J_{45}) - J_{04} + J_{34} + J_{15} - J_{25} + b_2 J_{67} + \cdots + b_{\frac{n-2}{2}} J_{nn+1}, \quad (39)$$

with norm

$$\begin{aligned} \xi_\mu \xi^\mu = & -b^2 - 4b(X_5(X_2 - V) + X_4(X_3 - U)) + (U - X_3)^2 + (V - X_2)^2 \\ & + b_2^2 (X_6^2 + X_7^2) + \cdots + b_{\frac{n-2}{2}}^2 (X_n^2 + X_{n+1}^2). \end{aligned} \quad (40)$$



This is everywhere spacelike if  $b = 0$ . For type  $III_{0(a)}$  we have

$$\xi = -aJ_{15} - J_{03} + J_{23} + b_1J_{67} + \cdots + b_{\frac{n-4}{2}}J_{nn+1}, \quad (41)$$

with norm

$$\xi_\mu \xi^\mu = (U + X_2)^2 + a^2(V^2 - X_5^2) + b_1^2(X_6^2 + X_7^2) + \cdots + b_{\frac{n-4}{2}}^2(X_n^2 + X_{n+1}^2), \quad (42)$$

for type  $III_{0(b)}$  we have

$$\xi = -J_{01} + J_{02} + b_1J_{45} + b_2J_{67} + \cdots + b_{\frac{n-2}{2}}J_{nn+1}, \quad (43)$$

with norm

$$\xi_\mu \xi^\mu = -(V + X_2)^2 + b_1^2(X_4^2 + X_5^2) + \cdots + b_{\frac{n-2}{2}}^2(X_n^2 + X_{n+1}^2), \quad (44)$$

and for type  $III_{0(c)}$  we have

$$\xi = -J_{13} + J_{23} + b_1J_{45} + b_2J_{67} + \cdots + b_{\frac{n-2}{2}}J_{nn+1}, \quad (45)$$

with norm

$$\xi_\mu \xi^\mu = (V + X_2)^2 + b_1^2(X_4^2 + X_5^2) + b_2^2(X_6^2 + X_7^2) + \cdots + b_{\frac{n-2}{2}}^2(X_n^2 + X_{n+1}^2). \quad (46)$$

This last case is spacelike everywhere away from the subspace  $V + X_2 = 0$ ,  $X_i = 0, i \geq 4$ , where it is null. Note that  $III_{0(c)}$  includes  $III_{0(a)}$  with  $a = 0$  as a special case. Finally, for type  $V$  we have

$$\xi = -J_{01} - J_{02} - J_{13} - J_{14} - J_{23} + J_{24} + b_1J_{67} + \cdots + b_{\frac{n-4}{2}}J_{nn+1}, \quad (47)$$

with norm

$$\begin{aligned} \xi_\mu \xi^\mu &= (V + X_2)^2 - 2X_4(U + X_3) \\ &\quad + b_1^2(X_6^2 + X_7^2) + \cdots + b_{\frac{n-4}{2}}^2(X_n^2 + X_{n+1}^2). \end{aligned} \quad (48)$$

When  $n$  is odd, the possibilities are slightly different:

- Type  $I$

$\mathbb{C}$	$\lambda_c (- - ++)$	$+ \lambda_0 (+)$	$+ \frac{n-3}{2} \lambda_i (+_1 +_2 \cdots +_{n-3}),$
$\mathbb{R}$	$2\lambda_r (- - ++)$	$+ \lambda_0 (+)$	$+ \frac{n-3}{2} \lambda_i (+_1 +_2 \cdots +_{n-3}),$
$\mathbb{I}$	$\frac{n+1}{2} \lambda_i (- - +_1 +_2 \cdots +_{n-1})$	$+ \lambda_0 (+),$	
$\mathbb{R}(0)$	$\lambda_r (-+)$	$+ \lambda_0 (-)$	$+ \frac{n-1}{2} \lambda_i (+_1 +_2 \cdots +_{n-1}).$

- Type *II*

$$\begin{aligned} \mathbb{R} \quad & \lambda_r^{II} (- - ++ ) \quad + \lambda_0 (+) \quad + \frac{n-3}{2} \lambda_i (+_1 +_2 \cdots +_{n-3}), \\ \mathbb{I} \quad & \lambda_i^{II} (- - ++ ) \quad + \lambda_0 (+) \quad + \frac{n-3}{2} \lambda_i (+_1 +_2 \cdots +_{n-3}). \end{aligned}$$

- Type *III*

$$\begin{aligned} \mathbb{I} \quad & \lambda_i^{III} (- - + + + ) \quad + \lambda_0 (+) \quad + \frac{n-5}{2} \lambda_i (+_1 +_2 \cdots +_{n-5}), \\ 0 \quad (a) \quad & \lambda_0^{III} (- + + ) \quad + \lambda_r (- + ) \quad + \frac{n-3}{2} \lambda_i (+_1 +_2 \cdots +_{n-3}), \\ 0 \quad (b) \quad & \lambda_0^{III} (- - + ) \quad + \frac{n-1}{2} \lambda_i (+_1 +_2 \cdots +_{n-1}), \end{aligned}$$

- Type *V*

$$\lambda_0^V (- - + + + ) \quad + \frac{n-3}{2} \lambda_i (+_1 +_2 \cdots +_{n-3}).$$

For the cases which occur for both even and odd  $n$ , the difference between the two cases is just that for either even or odd  $n$ , there is a direction which does not participate in the quotient; that is, they differ by a factor of  $\lambda_0$ . It is therefore not worth repeating the expressions for the Killing vectors in these cases for  $n$  odd. For the one new case, type  $I_{\mathbb{R}(0)}$ , the Killing vector is

$$\xi = aJ_{12} + b_1J_{34} + b_2J_{56} + \cdots + b_{\frac{n-1}{2}}J_{nn+1}, \quad (49)$$

with norm

$$\xi_\mu \xi^\mu = a^2(V^2 - X_2^2) + b_1^2(X_3^2 + X_4^2) + b_2^2(X_5^2 + X_6^2) + \cdots + b_{\frac{n-1}{2}}^2(X_n^2 + X_{n+1}^2). \quad (50)$$

For  $a = 0$ , this is the same as type  $I_{\mathbb{I}}$  with  $b_1 = 0$  in odd dimension. It is spacelike away from  $X_i = 0, i \geq 3$ , which is an axis where the Killing vector degenerates.

This completes the basic classification of different one-parameter subgroups of  $SO(2, n)$ , which is the central result of our paper. Most of the quotients determined by these Killing vectors will have causal pathologies, so they are not of great physical interest. The identification and description of the physically interesting cases is the subject of a companion paper [15].

To conclude this paper, we briefly describe how convenient coordinate systems can be defined on  $\text{AdS}_{n+1}$  based on the construction of the quotients out of invariant subspaces. These coordinate systems are quite useful in understanding the relation between quotients for different values of  $n$  and in working out their physics.

We have observed that the Killing vector describing each distinct type of quotient naturally decomposes into an  $SO(2, k)$  Killing vector, with  $k \leq 4$ , and a series of  $SO(2)$  rotations in independent planes. This decomposition can be made explicit

if we work in a suitable coordinate system. For most types,  $\xi$  can be decomposed in terms of an  $SO(2, 2)$  Killing vector acting on the coordinates  $U, V, X_2, X_3$  and rotations acting on the remaining  $X_i$  coordinates,  $i = 4, \dots, n+1$ . We can then construct a suitable coordinate system on  $\text{AdS}_{n+1}$  (for  $n \geq 3$ ) by writing

$$U = \cosh \chi \cosh \rho \cos t, \quad V = \cosh \chi \cosh \rho \sin t, \quad (51)$$

$$X_2 = \cosh \chi \sinh \rho \cos \phi, \quad X_3 = \cosh \chi \sinh \rho \sin \phi, \quad (52)$$

$$X_i = \sinh \chi x_i, \quad (53)$$

where  $i = 4, \dots, n+1$ , and  $x_i^2 = 1$ , so the metric on  $\text{AdS}_{n+1}$  is

$$\begin{aligned} ds^2 &= \cosh^2 \chi (-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\phi^2) + d\chi^2 + \sinh^2 \chi d\Omega_{n-3} \\ &= \cosh^2 \chi ds_{\text{AdS}_3}^2 + d\chi^2 + \sinh^2 \chi d\Omega_{n-3}. \end{aligned} \quad (54)$$

In this coordinate system, we can write  $\xi = \xi_3 + \xi_r$ , where  $\xi_3$  acts only on the  $\text{AdS}_3$  part, while the  $\xi_r$  is a rotation acting on the unit sphere  $S^{n-3}$ . Furthermore,  $\xi_3$  is precisely the Killing vector associated to the same type of quotient in the analysis of [12]. Similar coordinate systems can be introduced in the remaining two cases, writing  $\text{AdS}_{n+1}$  in terms of an  $\text{AdS}_4 \times S^{n-4}$  or  $\text{AdS}_5 \times S^{n-5}$  foliation. We will not repeat the details of the coordinate transformation, which are quite similar to the above case.

The coordinate system (54) also gives us an interesting description of the asymptotic boundary; taking the limit  $\chi \rightarrow \infty$  and conformally rescaling by a factor of  $e^{-2\chi}$ , we can describe the asymptotic boundary in terms of  $\text{AdS}_3 \times S^{n-3}$  coordinates;

$$ds_\Sigma^2 = (-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\phi^2) + d\Omega_{n-3}. \quad (55)$$

This description is related to the usual Einstein Static Universe (ESU) metric  $\mathbb{R} \times S^{n-1}$  on the conformal boundary of  $\text{AdS}_{n+1}$ ,

$$\tilde{ds}_\Sigma^2 = -dt^2 + d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 \theta d\Omega_{n-3}, \quad (56)$$

by a coordinate transformation  $\cosh \rho = 1/\cos \theta$  and a conformal rescaling  $ds_\Sigma^2 = \cosh^2 \rho \tilde{ds}_\Sigma^2$ . Hence, the  $\text{AdS}_3 \times S^{n-3}$  coordinates cover all of the  $S^{n-1}$  in the ESU except for one pole.

These coordinatizations show that the action of a given quotient on  $\text{AdS}_{n+1}$  can be simply expressed in terms of the action of the corresponding quotient on  $\text{AdS}_3$  (or  $\text{AdS}_4$  or  $\text{AdS}_5$ ) subspaces of the  $\text{AdS}_{n+1}$  together with rotations in an orthogonal sphere. In addition, the action of the quotient on the boundary of  $\text{AdS}_{n+1}$  for  $n > 2$  ( $n > 3$ ,  $n > 4$  respectively) is also expressed in terms of the action on the bulk of the lower-dimensional space. This observation will be used extensively in the study of the physics of these quotients in [15].

The main purpose of this paper was to explore the extension of the classification of one-parameter quotients of  $\text{AdS}_d$ , discussed in [12, 13] for the cases  $d = 3, 4$ , to the general case. This extension proved to be reasonably direct. Perhaps surprisingly,

there was little novelty in the general analysis; almost all the cases that appear for general  $d$  have appeared already for  $d = 3$  [12] or 4 [13]. The one exception, type  $III_{\mathbb{I}}$ , extends a particular quotient considered for the case  $d = 5$  in [11].

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